CHAPTER

1

INTRODUCTION

1.1 BACKGROUND

The finite element method is a numerical technique that gives approximate solutions to differential equations that model problems arising in physics and engineering. As in simple finite difference schemes, the finite element method requires a problem defined in geometrical space (or domain), to be subdivided into a finite number of smaller regions (a mesh).

In finite difference methods in the past, the mesh consisted of rows and columns of orthogonal lines (in computational space – a requirement now handled through coordinate transformations and unstructured mesh generators); in finite elements, each subdivision is unique and need not be orthogonal. For example, triangles or quadrilaterals can be used in two dimensions, and tetrahedra or hexahedra in three dimensions. Over each finite element, the unknown variables (e.g., temperature, velocity, etc.) are approximated using known functions; these functions can be linear or higher- order polynomial expansions in terms of the geometrical locations (nodes) used to define the finite element shape. In contrast to finite difference procedures (conventional finite difference procedures, as opposed to the finite volume method, which is integrated), the governing equations in the finite element method are integrated over each finite element and the contributions summed ("assembled") over the entire problem domain. As a consequence of this procedure, a set of finite linear equations is obtained in terms of the set of unknown parameters over the elements. Solutions of these equations are achieved using linear algebra techniques.

1.2 SHORT HISTORY

The history of the finite element method is particularly interesting, especially because the method has only been in existence since the mid-1950s. The early work on numerical solution of boundary-valued problems can be traced to the use of finite difference schemes; Southwell (1946) discusses the use of such methods in his book published in the mid-1940s. The beginnings of the finite element method actually stem from these early numerical methods and the frustration associated with attempting to use finite difference methods on more difficult, geometrically irregular problems (Roache, 1972).

Beginning in the mid 1950s, efforts to solve continuum problems in elasticity using small, discrete "elements" to describe the overall behavior of simple elastic bars began to appear. Argyris (1954) and Turner, et al. (1956) were the first to publish use of such techniques for the aircraft industry. Actual coining of the term "finite element" appeared in a paper by Clough (1960).

The early use of finite elements lay in the application of such techniques for structural-related problems. However, others soon recognized the versatility of the method and its underlying rich mathematical basis for application in non-structural areas. Zienkiewicz and Cheung (1965) were among the first to apply the finite element method to field problems (e.g., heat conduction, irrotational fluid flow, etc.) involving solution of Laplace and Poisson equations. Much of the early work on non-linear problems can be found in Oden (1972). Efforts to model heat transfer problems with complex boundaries are discussed in Huebner (1975); a comprehensive 3-D finite element model for heat conduction is described by Heuser (1972). Early application of the finite element technique to viscous fluid flow is given in Baker (1971).

Since these early works, rapid growth in usage of the method has continued since the mid 1970s. Numerous articles and texts have been published, and new applications appear routinely in the literature. Excellent reviews and descriptions of the method can be found in some of the earlier texts by Finlayson (1972), Desai (1979), Becker, et al. (1981), Baker (1983), Fletcher (1984), Reddy (1984), Segerlind (1984), Bickford (1990) and Zienkiewicz and Taylor (1989). A vigorous mathematical discussion is given in the text by Johnson (1987), and programming the finite element method is described by Smith (1982). A short monograph on development of the finite element method is given by Owen and Hinton (1980).

The underlying mathematical basis of the finite element method first lies with the classical Rayleigh-Ritz and variational calculus procedures introduced by Rayleigh (1877) and Ritz (1909). These theories provided the reasons why the finite element method worked well for the class of problems in which variational statements could be obtained (e.g., linear diffusion type problems). However, as interest expanded in applying the finite element method to more types of problems, the use of classical theory to describe such problems became limited and could not be applied (this is particularly evident in fluid-related problems).

Extension of the mathematical basis to non-linear and non-structural problems was achieved through the method of weighted residuals, originally conceived by Galerkin (1915) in the early 20th century. The method of weighted residuals was found to provide the ideal theoretical basis for a much wider basis of problems as opposed to the Rayleigh-Ritz method. Basically, the method requires the governing differential equation to be multiplied by a set of predetermined weights and the resulting product integrated over space; this integral is required to vanish. Technically, Galerkin's method is a subset of the general weighted residuals procedure, since various types of weights can be utilized; in the case of Galerkin's method, the weights are chosen to be the same as the functions used to define the unknown variables.

Galerkin and Rayleigh-Ritz approximations yield identical results whenever a proper variational statement exists and the same basis functions are used. By using constant weights instead of functions, the weighted residual method yields the finite volume technique. A more vigorous description of the method of weighted residuals can be found in Finlayson (1972). Recent descriptions of the method are discussed in Chandrupatla and Belegundu (2002), Liu and Quek (2003), Hollig (2003), Bohn and Garboczi (2003), Hutton (2004), Solin et al (2004), Reddy (2004), Becker (2004), and Ern and Guermond (2004).

Most practitioners of the finite element method now employ Galerkin's method to establish the approximations to the governing equations. The underlying theme in this book likewise follows Galerkin's method. The simplicity and richness of the method pays for itself as the user progresses into more complicated and demanding types of problems. Once this fundamental concept is grasped, application of the finite element method unfolds quickly.

1.3 ORIENTATION

This book is designed to serve as a simple introductory text and self-explanatory guide to the finite element method. Beginning with the concept of one-dimensional heat transfer (which is relatively easy to follow), the book progresses through two-dimensional elements to three-dimensional elements, ultimately ending with a discussion on various applications, including fluid flow. Particular emphasis is placed on the development of the one-dimensional element. All the principles and formulation of the finite element method can be found in the class of one-dimensional elements; extrapolation to two and three dimensions is straightforward.

Each chapter contains a set of example problems and some exercises that can be verified manually. In most cases, the exact solution is obtained from either inspection or an analytical equation. By concentrating on example problems, the manner and procedure for defining and organizing the requisite initial and boundary condition data for a specific problem becomes apparent. In the first few examples, the solutions are apparent; as the succeeding problems become progressively more involved, more input data must be provided.

For those problems requiring more extensive calculational effort (which becomes quickly discovered when dealing with matrices), a set of computer codes is available on the accompanying CD. These source codes are written in FORTRAN 95 (including several simple MATLAB, MATHCAD, and MAPLE routines) and run on WINDOWS-based PCs. The purpose of these codes is to illustrate simple finite element programming and to provide the reader with a set of programs that will assist in solving the examples and most of the exercises. The computer codes are fairly generic and have been written with the intention of instruction and ease of use. The reader may modify and optimize them as desired. The reader is advised to read the file called README.DOC on the CD for a discussion of the codes and execution procedures. Two additional codes are also available from the web (see http://www.ncacm.unlv.edu). One is written in C/C++ and the other is written in JAVA. Both permit 2-D heat transfer calculations to be run in real time under WINDOWS and on the WEB. These two codes include simple pre- and post processing of meshes and results.

A set of self executing files that use FEMLAB 3.1, a finite element code developed by COMSOL, is also included. FEMLAB is a fairly recent commercial finite element package, originally written to run with MATLAB, which is easy to use yet handles a wide variety of problems. The software can be used to solve 1-, 2-, and 3-D problems in structural analysis, heat transfer, fluid flow, and electrodynamics, and employs a rather elaborate, but easy to use mesh generator. The software also permits mesh adaptation (an upper end capability in recent commercial finite element packages that allows local mesh refinement in regions of steep gradients and high activity).

A discussion of the method of weighted residuals is given in Chapter 2. This chapter provides the underlying mathematical basis of the Galerkin procedure that is basic to the finite element method. Chapter 3 serves as the actual beginning of the finite element method, utilizing the one-dimensional element -- in fact, the entire framework of the method is presented in this chapter. Reinforcement of the basic concepts is achieved in Chapters 4 through 6 as the reader progresses through the class of two-dimensional elements. In Chapter 7, simple three-dimensional elements are discussed, utilizing a single element heat conduction problem with various boundary conditions, including radiation. Chapter 8 describes applications to solid mechanics and the role of multiple degrees of freedom (e.g., displacement in x and y) with example problems in two dimensions. Chapter 9 discusses applications to convective transport, using examples from potential flow and species dispersion. In Chapter 10, the reader is introduced to viscous fluid flow and the non-linear equations of fluid motion for an incompressible fluid. FEMLAB is particularly effective at solving fluid flow problems. The advanced book by Heinrich and Pepper (1999) discusses fluid flow in greater detail, and includes a 2-D penalty approach code and source listing for incompressible fluid flow.

The finite element method has essentially become the de facto standard for numerical approximation of the partial differential equations that define structural engineering, and is becoming widely accepted for a multitude of other engineering and scientific problems. Many of the commercial computer codes currently today are finite element based – even the finite volume computational fluid dynamics codes sold commercially employ mesh generators based on finite element unstructured mesh generation. It is the intent of this text to provide the reader with sufficient information and knowledge to begin application of the finite element method, and hopefully to instill an interest in advancing the state-of-the-art in more advanced studies.

Since the first edition of this book, there has been a proliferation of commercial codes for the finite element method, including many that are applicable to a wide range of problems. Recent introduction of the finite element method through generalized mathematical solvers, such as MATHCAD, MAPLE and MATLAB (and the supplementary software FEMLAB) have helped to spread the training and use of the method. The development of the finite element method using these mathematical symbolic systems is described in Pintur (1998), Portela and Charafi (2002) and Kattan (2003). A computer-based finite element analysis software package that runs on PCs and MacIntosh computers is VisualFEA/CBT, developed by Intuition Software (2002). This package permits up to 3000 nodes and runs structural, heat conduction, and seepage analysis.

1.4 CLOSURE

There are some interesting web sites that describe finite element methods and have codes that can be downloaded. It is recommended that the reader visit the following web sites for more detailed information regarding commercial solvers and mathematical software packages:

http://www.ncacm.unlv.edu
http://www.cfd-online.com/Resources/topics.html#fem

http://femcodes.nscee.edu

Web sites have a tendency to change locations and addresses over time. Performing a Google search on the subject finite elements will generate numerous web sites as well, many connected to universities and institutions around the world.

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